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The theory of hot white dwarfs of small and intermediate masses within the hybrid model

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Abstract. A new equation of state for hot white dwarfs of small and intermediate masses, which takes into account the contribution of an ideal relativistic electron subsystem, the contribution of a gas nuclear subsystem, and the light pressure was proposed. The internal structure of dwarfs was calculated within such model. It was shown that the proposed model agrees with the observed distribution of white dwarfs on the mass-radius plane. Based on the observed data on masses and radii of white dwarfs, there was solved the inverse problem – the determination of model parameters for the specific stars.

Key words: methods: analytical – stars: white dwarfs – stars: fundamental parameters – stars: interiors

1. Introduction

The discovery of white dwarfs at the beginning of the last century (Adams, 1915) gave a rise to the problem of energy sources and stability of stars, which have no thermonuclear reactions or occur only in the outer layers with a low intensity and therefore play a secondary role. According to the idea of Fowler (1926), the stability of white dwarfs is provided by the quantum effect – the degeneracy of the electron subsystem, when k_BT is much smaller than the electron energy at the Fermi surface. Chandrasekhar generalized this idea to the case of high densities, for which the electron subsystem is relativistic, and constructed the theory of cold white dwarfs based on the hydrostatic equilibrium equation (Chandrasekhar, 1931). The Chandrasekhar model is two-component: a completely degenerate ideal electron subsystem and a subsystem of nuclei, which is considered as a continuous classical environment. The equilibrium between the pressure of electron gas at $T = 0\,K$ and the gravitational compression, created by the nuclear subsystem provides the stability of a white dwarf. The white dwarf characteristics (mass, radius, and distribution of matter) are functions of two dimensionless parameters: the relativistic parameter $x_0 = \hbar(m_ec)^{-1}(3\pi^2 n(0))^{1/3}$ ($n(0)$ is the number density of electrons in the stellar center, m_e is the electron mass, c is the speed of light) and the chemical composition parameter $\mu_e = \langle A/z\rangle \approx 2.0$, where A is the mass number, z is the charge of the nucleus. The main conclusions from Chandrasekhar's theory are the peculiar unambiguous mass-radius dependence and the restriction on the maximal mass (the Chandrasekhar limit),

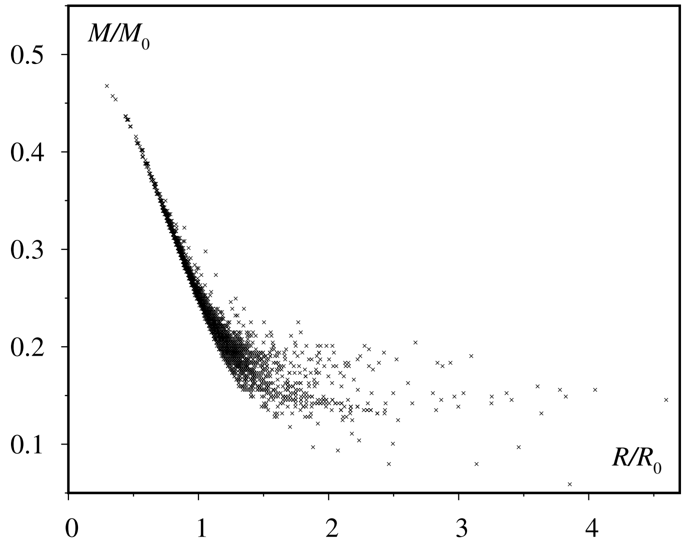
$$\begin{aligned} M_{\max} &= 2.01824\ldots \mu_e^{-2} M_0, \\ M_0 &= \left(\frac{3}{2}\right)^{1/2} \frac{1}{4\pi m_u^2} \left(\frac{hc}{G}\right)^{3/2} \approx 2.8866\ldots M_\odot, \end{aligned} \tag{1}$$

where m_u is the atomic mass unit. The restriction on the mass is confirmed by observations: for 110 years white dwarfs with masses $M > M_{\max}$ have not been detected. However, the mass-radius relation is satisfied only for white dwarfs of quite large masses ($M_{\max} > M \gtrsim 0.2 M_0$) with small radii,

$$R \lesssim R_0 = \left(\frac{3}{2}\right)^{1/2} \frac{1}{4\pi m_u m_e} \left(\frac{h^3}{cG}\right)^{1/2} \approx 1.1 \cdot 10^{-2} R_\odot. \tag{2}$$

This is clearly illustrated in Fig. 1*a*, constructed from the data of masses and radii of the observed white dwarfs from the catalog of Tremblay et al. (2011). The envelope dashed curve in Fig. 1*b* corresponds to the Chandrasekhar theory. The chains in Fig. 1*b* correspond to the observed white dwarfs with the same effective temperature, from which follow the conclusion about the important role of the finite temperature effects in the formation of white dwarfs' characteristics with small and intermediate masses. The problem of the internal structure of hot white dwarfs of small masses became relevant only after a wide variety of the characteristics of white dwarfs (masses, radii, effective temperatures, and luminosities) were revealed during the last 25 years with the help of space observatories. The Chandrasekhar model uses the equation of state of a completely degenerate ideal electron subsystem. From a modern view point, it is quite idealized and does not take into account the important factors of the white dwarfs' structure formation: interparticle interactions, axial rotation, magnetic fields and finite temperature effects. This is the basic model that explains the existence and stability of massive cold white dwarfs. It cannot explain the radiation of white dwarfs and the main details of their distribution on the mass-radius plane. Hot low-mass white dwarfs should not be considered correctly within the Chandrasekhar model, because their effective temperature reaches $(4 \div 8) \cdot 10^4\,K$, and the luminosity of some of them exceeds the luminosity of the Sun.

The problem of energy sources of white dwarfs was actively discussed in 1939-1952. Marshak (1940) followed the idea of gravitational compression, but did not investigate this mechanism. Schatzman proposed the idea of thermonuclear reactions in the surface layers of white dwarfs. A physically based idea, according to which white dwarfs emit a reserve of thermal energy accumulated in the past due to thermonuclear reactions, was proposed by Kaplan (1949, 1950). According to his idea, thermal energy is mainly concentrated in the nuclear subsystem, which can be considered as a classical ideal gas. The average value



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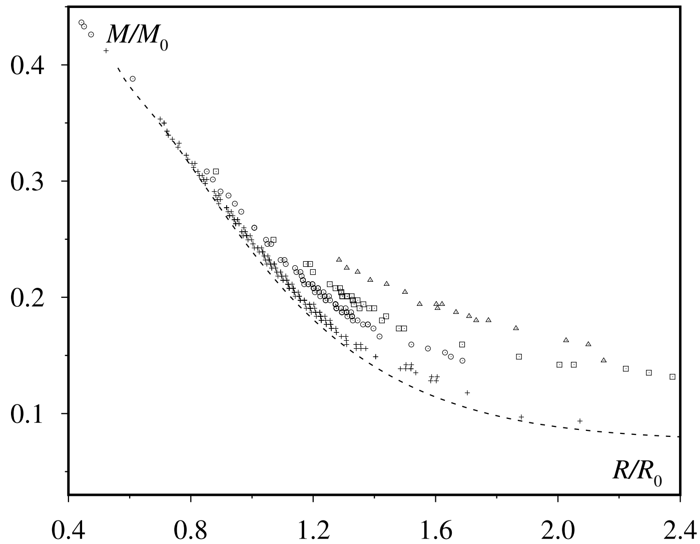

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Figure 1. The coordinates of white dwarfs of the spectral class DA with different temperatures on the mass-radius plane (data about masses and radii are taken from the catalog of Tremblay et al. (2011)). The chains correspond to the white dwarfs with close values of effective temperatures. The dashed curve is the mass-radius dependence for the cold white dwarfs.

of the kinetic energy of such model is

$$W_T \approx \frac{3}{2}\frac{k_B}{m_u\mu_n}\int\limits_V \rho(\mathbf{r})T(\mathbf{r})d\mathbf{r} \approx \frac{3}{2}\frac{k_B}{m_u\mu_n}MT, \tag{3}$$

where $\rho(\mathbf{r})$ is the density of matter, μ_n is the dimensionless molecular mass of the nucleus in the units m_u, $T \equiv \langle T(\mathbf{r})\rangle$ is the average value of temperature by the stellar volume. We can estimate the temperature value in the inner part of a white dwarf (which is considered nearly isothermal due to the electron mechanism of heat conduction) by the Schatzman (1947) formula

$$T = T_0\left(\frac{L/L_\odot}{M/M_\odot}\right)^{2/7}, \quad T_0 = 6.16\cdot 10^7\,K. \tag{4}$$

This expression is obtained by analyzing the system of structural equations in the peripheral region of a white dwarf (Shapiro & Teukolsky, 1983). According to Kaplan's idea, the luminosity is

$$L \approx -\frac{d}{dt}W_T \approx -\frac{3}{2}\frac{k_B}{m_u\mu_n}M\frac{dT}{dt}. \tag{5}$$

On the other hand, the luminosity can be determined from equation (4),

$$L = L_\odot\frac{M}{M_\odot}\left(\frac{T}{T_0}\right)^{7/2}. \tag{6}$$

Equating the right-hand sides of equalities (5) and (6), Kaplan obtained a differential equation that determines the dependence of the white dwarf's temperature on its age, or cooling time of a white dwarf (Kaplan, 1950)

$$\tau = \tau_0\left(\frac{M/M_\odot}{L/L_\odot}\right)^{5/7}, \quad \tau_0 = \frac{3}{5}\frac{k_BT_0}{m_u\mu_n}\frac{M_\odot}{L_\odot} \approx \frac{5}{\mu_n}\cdot 10^7\,\text{years}. \tag{7}$$

At the same time, it is assumed that the temperature of the white dwarf at the time of its formation was much higher than observed at the present time. For low-luminosity white dwarfs $L/L_\odot \approx 10^{-5}$, therefore the cooling time is of the order of 10^9 years. For hot white dwarfs with masses $M \approx 0.2M_0$ and effective temperatures $T_{\rm eff} \approx 5\cdot 10^4\,K$ the luminosity $L \approx L_\odot$, therefore the cooling time for them is close to 10^7 years. For bright white dwarfs of small masses with temperatures $T_{\rm eff} \approx (7\div 9)\cdot 10^4\,K$, the cooling time is of the order of $3\cdot 10^6$ years.

In Tab. 1 there are shown the data of masses, radii and luminosities of low-mass dwarfs with effective temperatures in the range $(2\div 9)\cdot 10^4\,K$. In this Table there are also shown the estimates of an average dimensionless temperature $T_* = k_BT/m_ec^2$ according to formula (4).

Table 1. The macroscopic characteristics of DA type white dwarfs from the catalog of Tremblay et al. (2011).

Number	R/R_0	M/M_0	$T_{\rm eff}$, K	T_*
3	1.8295	0.145498	23010	0.00669
111	1.91470	0.142034	25740	0.00787
104	2.3415	0.176676	72130	0.02692
106	2.87548	0.180140	84430	0.03605
158	3.82335	0.148962	78960	0.04149
266	1.74757	0.152426	28480	0.00821
377	2.11256	0.128177	20890	0.00671
688	1.95930	0.142034	27970	0.00876
592	2.42551	0.193997	86620	0.03297
921	1.29941	0.232104	54370	0.01288
928	2.27731	0.200926	88080	0.03209
2472	1.28454	0.232104	51770	0.01209
2531	1.34523	0.221711	50170	0.01213
2836	0.89115	0.322174	58380	0.01025
2929	1.24836	0.245961	60420	0.01397

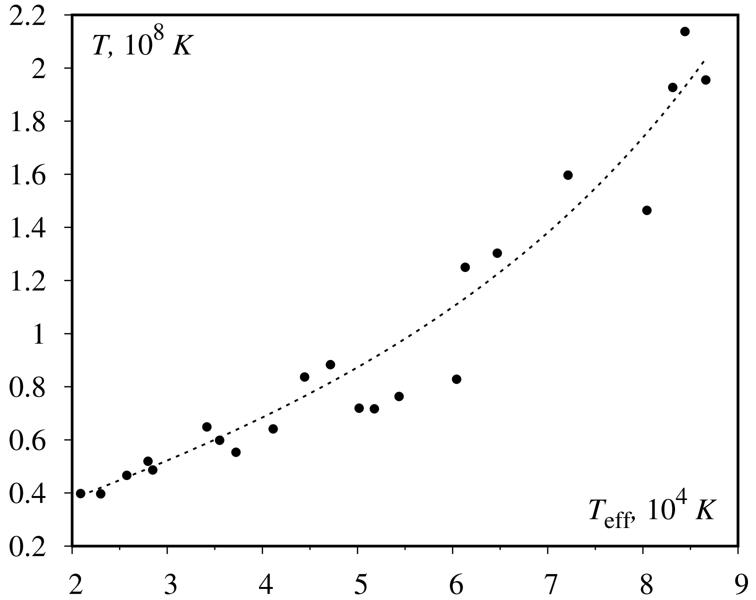

Figure 2. Dependence of the temperature T on the effective temperature $T_{\rm eff}$, calculated by formula (4) for the group of white dwarfs from the catalog of Tremblay et al. (2011).

It should be expected that due to the high thermal conductivity of white dwarfs, the temperature in the inner region is higher, the higher effective temperature. This is confirmed by Fig. 2, which shows the dependence "$T_{\rm eff} - T$", where $T_{\rm eff}$ is taken from the catalog of Tremblay et al. (2011), and T is calculated by formula (4). It would be incorrect to neglect the pressure of the nuclear subsystem at such high temperatures.

To clarify the elementary theory of cooling of white dwarfs as described above, we used a model in which the structure of the nuclear subsystem can vary from an ideal gas to a crystal lattice (Shapiro & Teukolsky, 1983). At the same time, the role of the electron subsystem was not taken into account, as in the work of Kaplan (1950).

Further development of this approach for the calculation of massive white dwarfs can be found in the work of Bisnovatyi-Kogan (1966), in which the temperature correction ($\sim T^2$) to the pressure of the electron subsystem is taken into account within the isothermal electron-nuclear model.

2. Model and equation of state

As it was shown from the calculation supplementing the Chandrasekhar model by the axial rotation (James, 1964) or interparticle interactions (Vavrukh et al., 2018), these have a little effect on the characteristics of cold massive dwarfs. High effective temperatures and observed distribution on the mass-radius plane indicate the important role of the finite temperature effects in the formation of hot white dwarfs structure. This is confirmed by approximate calculations within the generalized Chandrasekhar model, in which the electron subsystem is not completely degenerate (Vavrukh & Smerechynskyi, 2012, 2013).

The purpose of our work is to calculate the internal structure of specific observed white dwarfs within a simple spherically symmetric two-component model: a partially degenerate ideal electron subsystem + a nuclear subsystem which is considered as an ideal classical gas. Such model corresponds to the equation of state

$$P(\mathbf{r}) = P_e(\mathbf{r}) + P_n(\mathbf{r}) + P_{\rm ph}(\mathbf{r}). \tag{8}$$

where $P_e(\mathbf{r})$ is the partial pressure of the electron subsystem on the sphere of radius r,

$$P_n(\mathbf{r}) = \frac{k_B T(\mathbf{r})}{m_u \mu_n}\rho(\mathbf{r}) \tag{9}$$

is the partial pressure of the nuclear subsystem, where $T(\mathbf{r})$ is the local temperature and $\rho(\mathbf{r})$ is the density of matter. The term

$$P_{\rm ph}(\mathbf{r}) = \frac{a}{3}T^4(\mathbf{r}) \tag{10}$$

takes into account the photon pressure ($a = k_B^4(\hbar c)^{-3}\pi^2/15$). The correct description of the electron subsystem of a hot white dwarf requires the usage of two-phase or three-phase models: because in the inner region the electron subsystem is relativistic and deviation from the total degeneracy is small; in the periphery region the subsystem is degenerate, but not relativistic; near the star surface electrons can be considered as an ideal classical gas (Vavrukh & Smerechynskyi, 2012). However, the contributions of the peripheral region to the integral characteristics are small due to the low density. Therefore, to simplify the problem, we will assume that the electron gas is everywhere degenerate, and its chemical potential μ is positive. The terms $P_e(\mathbf{r})$ and $P_n(\mathbf{r})$ depend on the thermodynamic parameters $\rho(\mathbf{r})$ and $T(\mathbf{r})$. In order to simplify the problem, we will pass to one parameter – the local relativistic parameter

$$x(\mathbf{r}) = \frac{\hbar}{m_e c}\left[3\pi^2 n(\mathbf{r})\right]^{1/3}, \tag{11}$$

where $n(\mathbf{r})$ is the number density of electrons on the sphere of radius r, and therefore the density of matter

$$\rho(\mathbf{r}) = \frac{m_u \mu_e}{3\pi^2}\left(\frac{m_e c}{\hbar}\right)^3 x^3(\mathbf{r}). \tag{12}$$

According to the method of Eddington (1926), the sum of light pressure and gas pressure of the nuclear subsystem can be approximately represented in the form of a polytropic dependence

$$P_{\text{ph}}(\mathbf{r}) + P_n(\mathbf{r}) = K[\rho(\mathbf{r})]^{4/3}, \quad K = \left[\frac{1-\beta}{\beta^4}\right]^{1/3} \frac{\hbar c}{(m_u \mu_n)^{4/3}} \left(\frac{45}{\pi^2}\right)^{1/3}, \tag{13}$$

where the coefficient β determines the relative value of gas pressure ($\beta = P_n(\mathbf{r})[P_n(\mathbf{r}) + P_{\text{ph}}(\mathbf{r})]^{-1} = const$). Relations (9) – (13) determine the dependence between the matter density and temperature,

$$k_B T(\mathbf{r}) = \gamma \frac{\hbar c}{(m_u \mu_n)^{1/3}} \rho^{1/3}(\mathbf{r}) = \gamma \left(\frac{\mu_e}{\mu_n}\right)^{1/3} (3\pi^2)^{-1/3} m_e c^2 x(\mathbf{r}), \tag{14}$$

and the coefficient γ is the root of the equation

$$\frac{\pi^2}{45}\gamma^4 + \gamma = \alpha, \quad \alpha \equiv K(\hbar c)^{-1}(m_u \mu_n)^{4/3}. \tag{15}$$

According to relations (12) – (14)

$$P_n(\mathbf{r}) + P_{\text{ph}}(\mathbf{r}) = \frac{\pi m_e^4 c^5}{3h^3} \frac{8\alpha}{(3\pi^2)^{1/3}} \left(\frac{\mu_e}{\mu_n}\right)^{4/3} x^4(\mathbf{r}). \tag{16}$$

As it was known (Shapiro & Teukolsky, 1983), the equation of state of an ideal homogeneous relativistic subsystem of electrons at finite temperatures is written in the following parametric form

$$\left\{\begin{array}{l} P_e = \dfrac{8\pi}{3h^3}\int\limits_0^\infty dp\, p^3 \dfrac{dE_p}{dp}\, n_p, \\ \dfrac{N_e}{V} = \dfrac{8\pi}{h^3}\int\limits_0^\infty dp\, p^2\, n_p, \end{array}\right. \tag{17}$$

where

$$E_p = [(m_e c^2)^2 + p^2c^2]^{1/2} - m_e c^2, \;\; n_p = \left\{1 + \exp[(E_p - \mu)/k_B T]\right\}^{-1}. \tag{18}$$

The deviation from the absolute degeneracy can be considered small if $\mu(k_BT)^{-1} \gg 1$. The approximate calculations of integrals (17) at this basis and the following exception μ from the obtained system of algebraic equations allows us to represent P_e as follows

$$\begin{aligned} &P_e(x) = \frac{\pi m_e^4 c^5}{3h^3}\left\{\mathcal{F}_0(x) + \mathcal{F}_2(x) + \mathcal{F}_4(x) + \cdots\right\}, \\ &\mathcal{F}_0(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\ln[x + (1+x^2)^{1/2}]; \\ &\mathcal{F}_2(x) = \frac{4\pi^2 T_*^2}{3}\,\frac{x(2+x^2)}{(1+x^2)^{1/2}}; \\ &\mathcal{F}_4(x) = -\frac{\pi^4}{45}T_*^4\left[72 + 136x^2 + 77x^4 + 18x^6\right]x^{-3}(1+x^2)^{-3/2}; \ldots . \end{aligned} \tag{19}$$

Here $\mathcal{F}_0(x)$ determines the pressure of the absolute degenerate electron subsystem (at $T = 0$) and has the following asymptotics

$$\mathcal{F}_0(x) \to \left\{\begin{array}{ll} 2x^4 - 2x^2 + \cdots & \text{at } x \gg 1, \\ \dfrac{8}{5}x^5 - \dfrac{4}{7}x^7 + \cdots & \text{at } x \ll 1. \end{array}\right. \tag{20}$$

The dimensionless temperature is determined by the relation

$$T_* = k_B T(m_e c^2)^{-1}. \tag{21}$$

To obtain the coordinate dependence of the partial pressure of the electron subsystem in the white dwarfs' model, x should be replaced in formulae (19) and (20) by the local value of the relativistic parameter (11).

To simplify the problem, we model the coordinate dependence of temperature in expressions (19). The significant deviation from the absolute degeneracy of the electron subsystem occurs when $k_BT(r)$ is proportional to the local value of the Fermi energy,

$$k_B T(r) \approx A \cdot m_e c^2\left\{[1 + x^2(r)]^{1/2} - 1\right\}. \tag{22}$$

We select the coefficient A so that for $x(r) \gg 1$ the relation between $T(r)$ and $x(r)$ is given by expression (14). At this condition

$$T_*(r) \approx \alpha(3\pi^2)^{-1/3}\left(\frac{\mu_e}{\mu_n}\right)^{1/3}\Big\{(1+x^2(r))^{1/2}-1\Big\}. \tag{23}$$

At this modeling of the coordinate dependence of the dimensionless temperature, all terms $\mathcal{F}_2(x(r))$, $\mathcal{F}_4(x(r))$, ... have the same asymptotics relative to the $x(r)$ as the main term of expansion (see (20)), temperature corrections are small and series (19) coincides. At this $\mathcal{F}_2(x(r)) \sim \alpha^2$, $\mathcal{F}_4(x(r)) \sim \alpha^4$ and etc. We restrict ourselves to the quadratic approximation by $T_*(r)$, which makes it possible to depict the model equation of state in the following approximate form

$$\begin{aligned} P(\mathbf{r}) \cong \frac{\pi m_e^4 c^5}{3h^3}\Big\{&\mathcal{F}_0(x(\mathbf{r})) + \frac{8\alpha}{(3\pi^2)^{1/3}}\left(\frac{\mu_e}{\mu_n}\right)^{4/3} x^4(\mathbf{r}) + \\ &+ 4\alpha^2\pi^{2/3}3^{-5/3}\left(\frac{\mu_e}{\mu_n}\right)^{2/3} f(x(\mathbf{r}))\Big\}, \end{aligned} \tag{24}$$

where

$$f(x(\mathbf{r})) = x(\mathbf{r})[2+x^2(\mathbf{r})](1+x^2(\mathbf{r}))^{-1/2}\big\{[1+x^2(\mathbf{r})]^{1/2}-1\big\}^2, \tag{25}$$

and α is the model parameter. Due to the presence of multiplier α^2, the third term in the curly bracket of formula (24) is small, which allows us to take it into account by the approximate method. As it is easy to see, the ratio $f(x)/\mathcal{F}_0(x)$ in the interval $0 \le x \le 1$ is a monotonously decreasing function that very weakly depends on x and varies from the value 0.3125 at $x = 0$ to the value 0.2959 at $x = 1$. Therefore, the sum of the first and third terms in the curly bracket of formula (24) with sufficiently high accuracy can be written in the form

$$\mathcal{F}_0(x(\mathbf{r}))\left\{1 + 1.216\alpha^2\pi^{2/3}3^{-5/3}\left(\frac{\mu_e}{\mu_n}\right)^{2/3}\right\} \equiv \mathcal{F}_0(x(\mathbf{r}))(1+B), \tag{26}$$

replacing the fraction $f(x)/\mathcal{F}_0(x)$ with its average value 0.304 on the interval $0 \le x \le 1$. Here

$$B = 1.216\alpha^2\pi^{2/3}3^{-5/3}\left(\frac{\mu_e}{\mu_n}\right)^{2/3}. \tag{27}$$

3. Equilibrium equation

The internal structure of a white dwarf is determined by the equilibrium equation

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r})\nabla\Phi(\mathbf{r}), \tag{28}$$

where

$$\Phi(\mathbf{r}) = -G \int\limits_V \frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{29}$$

is the gravitational potential on the sphere of radius r, created by the distribution of matter. Using equation of state (24) in approximation (26) and relation (12), let us reduce equation (28) to

$$(1+B)\frac{x(\mathbf{r})\nabla x(\mathbf{r})}{[1+x^2(\mathbf{r})]^{1/2}} + C\nabla x(\mathbf{r}) = -\frac{m_u\mu_e}{m_e c^2}\nabla\Phi(\mathbf{r}), \tag{30}$$

where

$$C = \frac{4\alpha}{(3\pi^2)^{1/3}}\left(\frac{\mu_e}{\mu_n}\right)^{4/3}. \tag{31}$$

Using the identity

$$\frac{x\nabla x}{[1+x^2]^{1/2}} = \nabla\big\{[1+x^2]^{1/2}-1\big\}, \tag{32}$$

acting by operator ∇ on equation (30) and taking into account that $\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$, we obtain the differential equation for the local value of the relativistic parameter

$$(1+B)\Delta\big\{[1+x^2(\mathbf{r})]^{1/2} - 1 + \tilde{C}x(\mathbf{r})\big\} = -\frac{32\pi^2 G}{3(hc)^3}\left[\frac{m_e m_u \mu_e}{c^2}\right]^2 x^3(\mathbf{r}), \tag{33}$$

where $\tilde{C} = C/(1+B)$. It follows from formulae (27) and (31) that $B = 1.216\, C^2\pi^2(48)^{-1}(\mu_n/\mu_e)^2$, and $\tilde{C} = C\{1+1.216\, C^2\pi^2(48)^{-1}(\mu_n/\mu_e)^2\}^{-1}$. We consider a white dwarf without axial rotation i. e. beings spherically symmetric, and its chemical composition is spatially homogeneous. Therefore, in our model there appear four dimensionless parameters: $x_0 \equiv x(0)$ (the relativistic parameter in the stellar center), C, μ_e and μ_n which for the specific white dwarf are constants.

In order to numerically solve the equilibrium equation, it is convenient to pass from the variables $(r,\, x(r))$ to the dimensionless variables

$$\xi = \frac{r}{\lambda}, \quad y(\xi) = \varepsilon_0^{-1}\big\{[1+x^2(r)]^{1/2} - 1 + \tilde{C}x(r)\big\}, \tag{34}$$

where

$$\varepsilon_0 = \varepsilon_0(x_0, \tilde{C}) = (1+x_0^2)^{1/2} - 1 + \tilde{C}x_0, \tag{35}$$

and λ is the length scale. If we determine it from the condition

$$\frac{32\pi^2 G}{3(hc)^3}\left\{\frac{m_e m_u \mu_e}{c^2}\lambda\varepsilon_0\right\}^2 = 1+B \cong 1 + 1.216\,\frac{\pi^2}{48}\tilde{C}^2\left(\frac{\mu_n}{\mu_e}\right)^2, \tag{36}$$

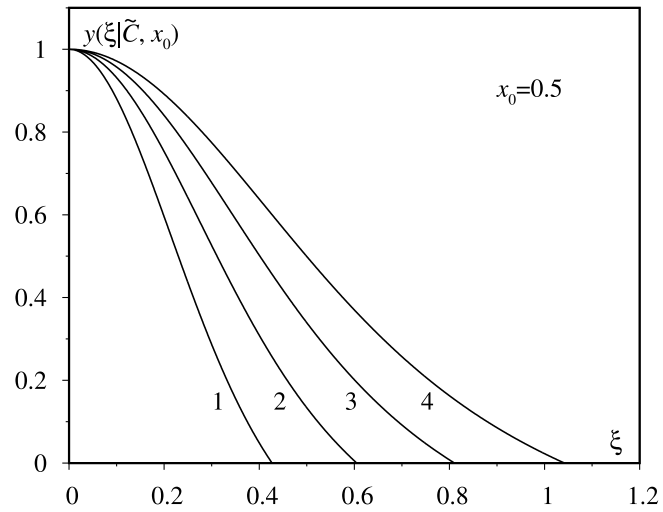

Figure 3. The solution of the equilibrium equation at the fixed value of $x_0 = 0.5$ for different values of the parameter $\tilde{C}$. Curve 1 corresponds to the standard model at $\tilde{C} = 0$, curve 2 – $\tilde{C} = 0.05$, 3 – $\tilde{C} = 0.1$, 4 – $\tilde{C} = 0.15$.

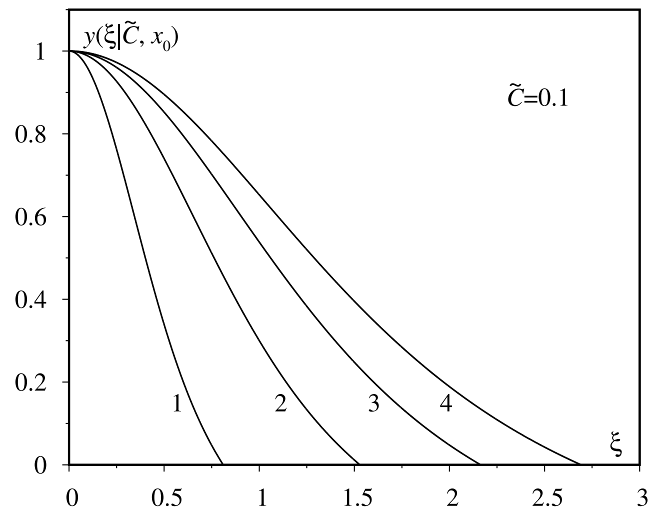

Figure 4. The solution of the equilibrium equation at the fixed value of $\tilde{C} = 0.1$ for different values of the parameter x_0. Curve 1 corresponds to $x_0 = 0.5$, curve 2 – $x_0 = 1.0$, 3 – $x_0 = 1.5$, 4 – $x_0 = 2.0$.

and express $x(r)$ through $y(\xi)$ using definition (34), then the equation for the function $y(\xi)$ takes the form

$$\Delta_\xi y(\xi) = -x^3(\xi, \tilde{C}),$$
$$x(\xi, \tilde{C}) = \frac{x(r)}{\varepsilon_0} \equiv \left\{\varepsilon_0(1-\tilde{C}^2)\right\}^{-1}\left\{[\tilde{C}^2 + (\varepsilon_0 y)^2 + 2\varepsilon_0 y]^{1/2} - \tilde{C}(1+\varepsilon_0 y)\right\}. \quad (37)$$

This equation corresponds to the boundary condition $y(0) = 1$, as well the condition $dy/d\xi = 0$ at $\xi = 0$, which ensures the non-singular nature of the solution in the white dwarf center. As it is easy to see, in the limit $\tilde{C} \to 0$ the right-hand side of equation (37) equals $(y^2(\xi) + 2\varepsilon_{00}^{-1} y(\xi))^{3/2}$, as in the Chandrasekhar model, where $\varepsilon_{00} \equiv \varepsilon_0(x_0, 0)$. A set of solutions of the two-parametric equation (37), found by the numerical method for several values of parameters $\tilde{C}$ and x_0 is illustrated in Figs. 3 and 4.

In accordance with relations (34) the radius of the white dwarf is

$$R(x_0, \mu_e, \tilde{C}) = \lambda \xi_1(x_0, \tilde{C}) = R_0 \, \frac{\xi_1(x_0, \tilde{C})}{\mu_e \varepsilon_0(x_0, \tilde{C})}(1+B)^{1/2}, \quad (38)$$

where $\xi_1(x_0, \tilde{C})$ is the root of equation $y(\xi|x_0, \tilde{C}) = 0$, and R_0 is determined by formula (2), $\lambda(C)$ is the root of equation (36), and parameters C and $\tilde{C}$ are related by

$$2\gamma C = \frac{1}{\tilde{C}} - \sqrt{\frac{1}{\tilde{C}^2} - 4\gamma}, \quad (39)$$

where $\gamma = \pi^2(48)^{-1}(\mu_n/\mu_e)^2$. The stellar mass is determined by the solution of equation (37),

$$M(x_0, \mu_e, \tilde{C}) = \frac{M_0}{\mu_e^2} \, \mathcal{M}(x_0, \tilde{C})(1+B)^{3/2}, \quad (40)$$

where

$$\mathcal{M}(x_0, \tilde{C}) = \int_0^{\xi_1(x_0, \tilde{C})} \xi^2 x^3(\xi, \tilde{C}) \, d\xi. \quad (41)$$

Dependence of the dimensionless radius on the model parameters is illustrated in Fig. 5. Herewith, the dashed curve corresponds to the standard Chandrasekhar model ($\tilde{C} = 0$). The dimensionless white dwarf mass $\mathcal{M}(x_0, \tilde{C})$ is illustrated in Fig. 6 as a function of x_0 for different values of the parameter $\tilde{C}$. Relations (14) and (15) give us an opportunity to determine the temperature change along the radius

$$T(\xi, \tilde{C}) = \gamma \left(\frac{\mu_e}{\mu_n}\right)^{1/3} \frac{m_e c^2}{(3\pi^2)^{1/3} k_B} x(\xi, \tilde{C}). \quad (42)$$

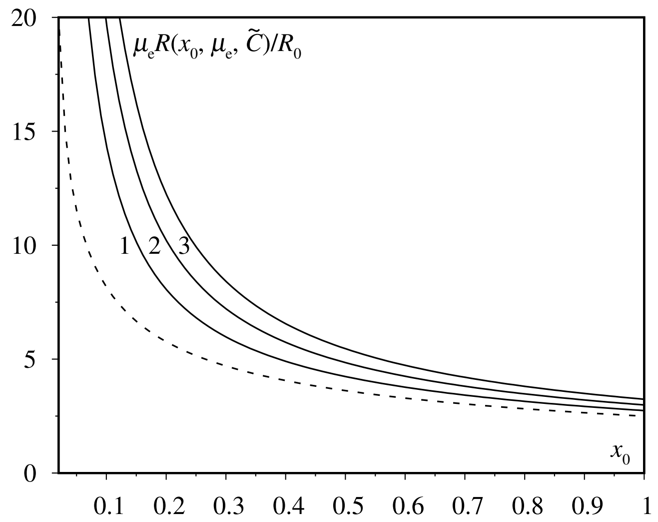

Figure 5. Dependence of the dimensionless radius on the relativistic parameter x_0 for different values of the parameter $\tilde{C}$. The dashed curve corresponds to the standard Chandrasekhar model with $\tilde{C} = 0$, curve 1 – $\tilde{C} = 0.05$, 2 – $\tilde{C} = 0.1$, 3 – $\tilde{C} = 0.15$.

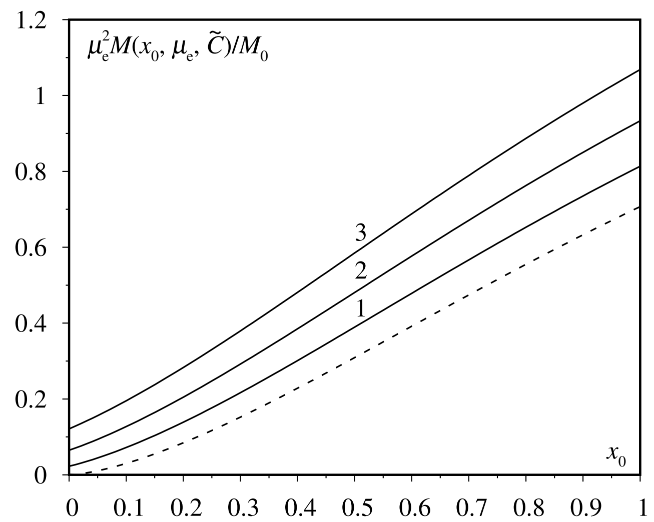

Figure 6. Dependence of the dimensionless mass $\mathcal{M}(x_0, \tilde{C})$ on the relativistic parameter x_0 for different values of the parameter $\tilde{C}$ (notations are the same as in the previous Figure).

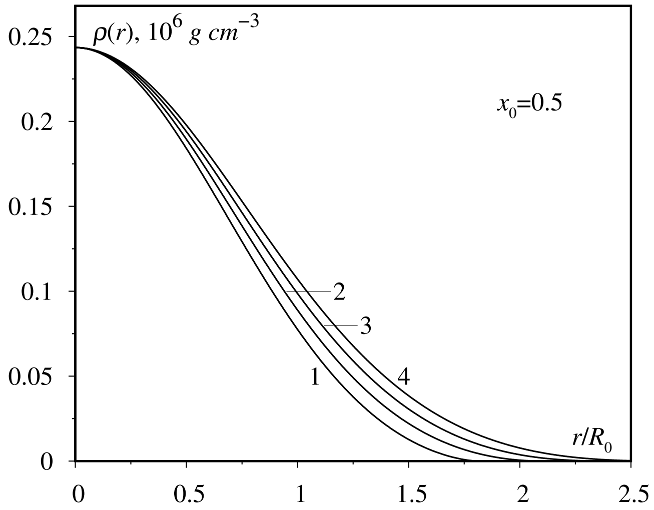

Figure 7. The distribution of matter density along the radius at the fixed value of $x_0 = 0.5$ for different values of the parameter $\tilde{C}$. Curve 1 corresponds to $\tilde{C} = 0$, 2 – $\tilde{C} = 0.05$, 3 – $\tilde{C} = 0.1$, 4 – $\tilde{C} = 0.15$.

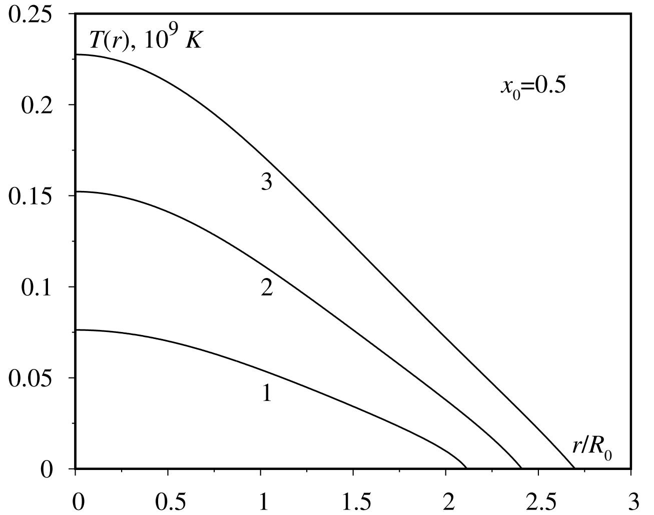

Figure 8. The distribution of temperature along the radius at the fixed value of $x_0 = 0.5$ for different values of the parameter $\tilde{C}$. Curve 1 corresponds to $\tilde{C} = 0.05$, 2– $\tilde{C} = 0.1$, 3 – $\tilde{C} = 0.15$.

To illustrate the capabilities of our approach in Figures 7 and 8 we show how both density and temperature vary with the radius.

The crosses in Fig. 9 depict the location of observed white dwarfs (from the catalog of Tremblay et al. (2011)) on the mass-radius plane. Curves in this figure depict the dependence between the mass and the radius of white dwarfs in our model. They were constructed from relations (38) – (41) for several values of the parameter $\tilde{C}$ for a range of the relativistic parameter ($0.3 \leq x_0 \leq 0.8$). Herewith, we used the value $\mu_e = 2$, as well as $\mu_n = 4$ (that corresponds to helium white dwarfs). As it is shown in Figure, in the selected region of the change of the parameter $\tilde{C}$ our model can describe the distribution of observed hot white dwarfs of small and intermediate masses on the mass-radius plane. This is impossible to achieve in the Chandrasekhar model, which corresponds to the lower curve in Figure with $\tilde{C} = 0$. The standard model satisfactorily describes only the distribution of white dwarfs with sufficiently large masses and radii $R(x_0, \tilde{C}) \lesssim R_0$.

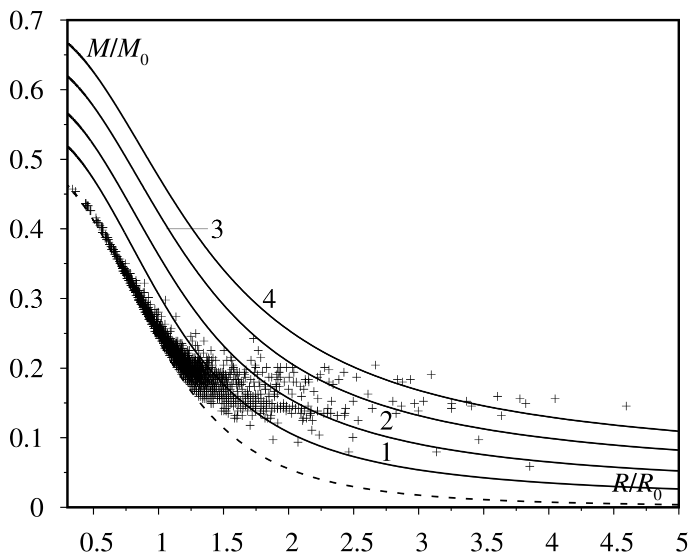

Figure 9. The mass-radius relation for different values of the parameter $\tilde{C}$. The dashed curve corresponds to the standard model ($\tilde{C} = 0$), curve 1 – $\tilde{C} = 0.05$, 2 – $\tilde{C} = 0.1$, 3 – $\tilde{C} = 0.15$, 4 – $\tilde{C} = 0.19$. The crosses correspond to the observed data from the catalog of Tremblay et al. (2011).

To calculate the internal structure of the observed white dwarfs, it is necessary first to solve the inverse problem of the theory – the determination of model parameters for a specific white dwarf based on the observed data. To demonstrate the algorithm for such calculation, we consider Fig. 9 from which it can be seen that a theoretical mass-radius curve can always be drawn through a point with the given observed mass M and radius R with the certain parameter value $\tilde{C}$. This is the parameter value $\tilde{C}$ for a chosen white dwarf. Putting instead $R(x_0, \tilde{C})$ and $M(x_0, \tilde{C})$ the observed R and M in relations (38) and (40), we obtain the system of equations for the parameters x_0 and μ_e

$$\begin{aligned} \frac{R}{R_0}\frac{\varepsilon_0(x_0,\tilde{C})}{\xi_1(x_0,\tilde{C})} &= \frac{1}{\mu_e}(1+B)^{1/2}, \\ \frac{M}{M_0} &= \frac{1}{\mu_e^2}\mathcal{M}(x_0,\tilde{C})(1+B)^{3/2}. \end{aligned} \tag{43}$$

Excluding μ_e, we obtain the equation

$$\frac{M}{M_0}\left(\frac{R_0}{R}\right)^2\left(\frac{\xi_1(x_0,\tilde{C})}{\varepsilon_0(x_0,\tilde{C})}\right)^2 = \mathcal{M}(x_0,\tilde{C})(1+B)^{1/2} \tag{44}$$

for finding the relativistic parameter x_0, after that the parameter μ_e is determined by the first of equations (43).

Since the parameter B plays the role of correction, it makes sense to consider a simplified version of the model with absolute degeneracy of the electron subsystem, that corresponds to $B = 0$ (or $\tilde{C} = C$). The model parameters for the group of white dwarfs from the catalog Tremblay et al. (2011) calculated in such way are shown in Tab. 2. Using relations (12) and (14), we calculated the density of matter and temperature in the center of these white dwarfs $\rho_c = \rho(0)$ and $T_c = T(0)$. These values are also shown in Tab. 2. As we can see, the temperature T_c^0 calculated by formula (4) is almost $(1.5 - 3)$ times smaller than the temperature T_c obtained by formula (42). This cannot be a criterion, because formula (4) is approximated and corresponds to the isothermal model.

In order to evaluate the role of incomplete degeneracy of the electron subsystem we considered a group of white dwarfs that corresponds to $C = 0.15$ in Tab. 2. At $B = 1.216\,(48)^{-1}\pi^2 C^2(\mu_n/\mu_e)^2$ these dwarfs correspond to the mass-radius curve with $\tilde{C} = 0.142$. Model parameters x_0 and μ_e are determined from the system of equations (43), as well as calculated on this basis $\rho_c(0)$ and $T_c(0)$ are shown in Tab. 3. From the comparison of Tabs. 2 and 3 it follows that the influence of incomplete degeneracy of the electron subsystem is weak and leads to small changes of parameters and characteristics of the model. In particular, the relative decreasing of the relativistic parameter x_0 has an order of 0.6%, decreasing ρ_c corresponds to 2%, and the decreasing T_c is close to 5%.

4. Discussion

According to Fowler's idea, the stability of white dwarfs ensured by the pressure of the degenerate electron subsystem is an extrapolation of the real thermodynamical state of these objects in the case of low temperatures or high pressures.

Table 2. The characteristics and the model parameters for the group of white dwarfs of spectral class DA from the catalog Tremblay et al. (2011) at $B = 0$ (T_c^0 approximately corresponds to equation (4)).

Number	R/R_0	M/M_0	x_0	μ_e	ρ_c, 10^6 g cm^{-3}	T_c, $10^9\,K$	T_c^0, $10^9\,K$
			$C = 0.1$				
2270	3.13742	0.079677	0.346458	2.03348	0.080993	0.105513	0.048904
3026	2.09943	0.142034	0.603049	2.00554	0.427125	0.183657	0.069994
441	1.88942	0.166283	0.703784	1.99963	0.678915	0.214336	0.090419
2581	1.67358	0.197461	0.841388	1.99714	1.160080	0.256243	0.106619
			$C = 0.15$				
2922	2.82645	0.135105	0.472760	1.99620	0.205788	0.215153	0.144974
1246	2.64303	0.142034	0.508990	2.01176	0.256818	0.231642	0.146958
2606	2.49710	0.152426	0.548329	2.00757	0.321086	0.249545	0.141763
2090	2.15203	0.183604	0.668973	1.99890	0.583074	0.304450	0.148608
2507	1.98346	0.200926	0.744181	2.00258	0.802663	0.338677	0.157729
			$C = 0.19$				
2017	3.63580	0.131641	0.387910	1.99912	0.113682	0.222433	0.069164
397	3.25419	0.148962	0.444555	1.98260	0.171109	0.254914	0.211635
592	2.42551	0.193997	0.626813	1.99922	0.479636	0.359423	0.199981

Chandrasekhar's theory is based on this idea. The position of observed white dwarfs on the mass-radius plane indicates the different ages of these stars and reflects their evolution. The generalized theory of white dwarfs can be built by generalization of Fowler's idea, using a more general form of the equation of state.

The simplest variant of such approach is proposed in our article to describe hot white dwarfs, whose nuclear subsystem can be considered as a classical gas. We have also approximately taken into account the incomplete degeneracy of the electron subsystem. The electron subsystem in our work is treated almost in the same way as in the Fowler-Chandrasekhar model, but the nuclear subsystem is treated as in the theory of main sequence normal stars. Since the nuclear subsystem contributes to the total internal pressure, the mass and radius of the white dwarf in this model are greater than in the standard Chandrasekhar model that corresponds to the observed data. The mass-radius curves calculated by us

Table 3. The characteristics and the model parameters for the group of white dwarfs of spectral class DA from the catalog Tremblay et al. (2011) at $B \neq 0$ ($\tilde{C} = 0.142$).

Number	R/R_0	M/M_0	x_0	μ_e	ρ_c, 10^6 g cm^{-3}	T_c, $10^9\,K$
2922	2.82645	0.135105	0.469604	1.99206	0.201694	0.202473
1246	2.64303	0.142034	0.505594	2.00879	0.251712	0.217990
2606	2.49710	0.152426	0.544676	2.00578	0.314711	0.234840
2090	2.15203	0.183604	0.664536	2.00011	0.571548	0.286519
2507	1.98346	0.200926	0.739257	2.00529	0.786835	0.318735

agree with the distribution of observed white dwarfs on the mass-radius plane. Herewith, the calculated radii of white dwarfs can exceed the corresponding values in Chandrasekhar's theory by 2-3 times.

The Fowler-Chandrasekhar model has dimensionless parameters x_0 and μ_e. The parameter $\tilde{C}$ reflects the partial pressure of the nuclear subsystem. We determine this parameter for a specific white dwarf based on the condition that the theoretically calculated mass-radius curve for a given $\tilde{C}$ passes through the point corresponding to the observed mass and radius of the white dwarf. After that the parameters x_0 and μ_e are determined from relations (43). This allows us to calculate all characteristics for a specific white dwarf (the distribution of matter and the temperature along the radius, moment of inertia, total energy and etc.).

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