Contrib. Astron. Obs. Skalnaté Pleso **53/2**, 16 – 30, (2023)
https://doi.org/10.31577/caosp.2023.53.2.16

# Component $x$ of the gravitational acceleration in general relativity and concept of mass

**L. Neslušan**

*Astronomical Institute of the Slovak Academy of Sciences*
*059 60 Tatranská Lomnica, The Slovak Republic, (E-mail: ne@ta3.sk)*

Received: September 5, 2023; Accepted: October 16, 2023

**Abstract.** In general relativity, the acceleration of a test particle in the vicinity of a massive object should be calculated by using the equation of geodesic. The specific formula for the acceleration is, however, well-known only in the Schwarzschild coordinates. Here, we present this formula in the frame with the spatial part identical to the common rectangular coordinate frame. The orientation of the acceleration and identification of physical quantities with the general integration constants can better be discerned in the latter. We emphasize that the gravitational acceleration in general relativity, for system of particles being in rest, consists of two terms. The first term is identical with the acceleration derived on the basis of Newton's gravitational law. The second term, having a smaller size above the event horizon, is a repulsive contribution to the total, attractive, gravity. The relativistic formula implies that the gravity should be repulsive below the event horizon, because the second term dominates in this region.

**Key words:** gravitational acceleration – general relativity – $x$-$y$-$z$ coordinate frame – repulsive contribution to gravity

## 1. Introduction

In Einstein's theory of general relativity (Einstein, 1915, 1916), the gravitational acceleration of a test particle (TP, hereafter) in the vicinity of other, massive, particle (MP) is given by the equation of geodesic (EoG). To calculate the acceleration, it is necessary to know the metrics of space-time generated by the MP in which the TP is situated. When the MP is a point-like particle, the outer Schwarzschild metrics (OSM) (Schwarzschild, 1916) is relevant.

The metric tensor characterizing the OSM is well known in the Schwarzschild coordinate system $\mathrm{O}(r, \vartheta, \varphi, ct)$ ($t$ is time and $c$ is the speed of light in vacuum), i.e. with the spatial part identical with the common spherical coordinate system $\mathrm{O}(r, \vartheta, \varphi)$. The components of the metric tensor enable us to calculate the size of the acceleration in the frame with the origin identical with the position of the point-like MP. Namely, the acceleration depends only on the $g_{rr}$ and $g_{tt}$ components of metric tensor, which both depend only on the radial distance from the MP, $r$.

However, sometimes we need to calculate the acceleration in a specific direction, in which its orientation is seen better than in the spherical coordinate frame. In this work, we derive the formula for the acceleration in the coordinate frame O($x$,$y$,$z$,$\tau$), i.e. the frame with the spatial part identical with the rectangular coordinate frame O($x$,$y$,$z$). We denoted $\tau = ct$. The general formula would be very large and, thus, difficult. Because of this reason, we adopt the assumption reducing the formula: we deal with the case when both MP and TP are in rest and, moreover, the MP is situated in the origin of the O($x$,$y$,$z$) frame.

In the Euclidean geometry, the transformation between the spherical and rectangular coordinates is trivial. However, one can doubt if this is also valid in a curved space-time of general relativity. In this paper, we explicitly make the transformation. Although it does not give us any new result, the demonstration of how the transformation can be performed might be useful. Perhaps, it can also serve as an exercise in teaching, in a basic course on general relativity.

Beside the explicit derivation of the transformation, we are interested in a comparison between the relativistic and Newtonian formulas for the gravitational acceleration, since this comparison can help us to understand a relation between the integration constants in the Schwarzschild solution of the field equations and mass.

## 2. Transformation of coordinate frames

The coordinates in the common spherical frame, O($r, \vartheta, \varphi$), are related to their counterparts in the rectangular frame, O($x$,$y$,$z$), as

$$x = r \cos\varphi \sin\vartheta, \tag{1}$$

$$y = r \sin\varphi \sin\vartheta, \tag{2}$$

$$z = r \cos\vartheta. \tag{3}$$

Based on these relations, the transformation relations from O($r, \vartheta, \varphi$) to O($x$,$y$,$z$) are

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{4}$$

$$\sin\vartheta = \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}}, \qquad \cos\vartheta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{5}$$

$$\sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}, \qquad \cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}. \tag{6}$$

In view of the calculation of derivatives of these coordinates, it is more useful to consider the relations

$$\tan\vartheta = \frac{\sqrt{x^2 + y^2}}{z}, \tag{7}$$

$$\tan\varphi = \frac{y}{x}, \tag{8}$$

or

$$\vartheta = \arctan \frac{\rho}{z}, \tag{9}$$

$$\varphi = \arctan \frac{y}{x}. \tag{10}$$

In relation (9), we used denotation

$$\rho = \sqrt{x^2 + y^2}. \tag{11}$$

Further, we also denote

$$R = \sqrt{x^2 + y^2 + z^2}. \tag{12}$$

Although $r = R$ formally symbol $R$ is not a variable; it is only a denotation of the square root of the sum of $x$, $y$, and $z$, all squared.

One can easily find that the partial derivatives of the spherical coordinates in respect to their rectangular counterparts are

$$\frac{\partial r}{\partial x} = \frac{x}{R}, \tag{13}$$

$$\frac{\partial \vartheta}{\partial x} = \frac{xz}{R^2 \rho}, \tag{14}$$

$$\frac{\partial \varphi}{\partial x} = -\frac{y}{\rho^2}, \tag{15}$$

$$\frac{\partial r}{\partial y} = \frac{y}{R}, \tag{16}$$

$$\frac{\partial \vartheta}{\partial y} = \frac{yz}{R^2 \rho}, \tag{17}$$

$$\frac{\partial \varphi}{\partial y} = \frac{x}{\rho^2}, \tag{18}$$

$$\frac{\partial r}{\partial z} = \frac{z}{R}, \tag{19}$$

$$\frac{\partial \vartheta}{\partial z} = -\frac{\rho}{R^2}, \tag{20}$$

$$\frac{\partial \varphi}{\partial z} = 0. \tag{21}$$

If we assume both TP and MP being in rest, then

$$\frac{\partial t}{\partial x} = 0, \tag{22}$$

$$\frac{\partial t}{\partial y} = 0, \tag{23}$$

$$\frac{\partial t}{\partial z} = 0, \tag{24}$$

$$\frac{\partial (ct)}{\partial \tau} = 1. \tag{25}$$

In the Schwarzschild coordinates with the spatial part being the spherical coordinates, only the diagonal components of the metric tensor are non-zero, i.e.

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix} \equiv \begin{pmatrix} g_{rr} & 0 & 0 & 0 \\ 0 & g_{\vartheta\vartheta} & 0 & 0 \\ 0 & 0 & g_{\varphi\varphi} & 0 \\ 0 & 0 & 0 & g_{tt} \end{pmatrix}. \tag{26}$$

This tensor can be transformed to its counterpart in the O($x$,$y$,$z$,$\tau$) coordinate frame by using a well-known transformation formula

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu} g_{\alpha\beta}. \tag{27}$$

The result of the transformation is given in Appendix A.

## 3. $x$-component of acceleration

As we mentioned in Sect. 1, we consider two particles, TP and MP. The MP is situated in the origin of the coordinate frame. The acceleration of the TP at distance $x$ from the MP is given by the EoG,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0, \tag{28}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols and $ds$ is the line element in the four-dimensional space-time. Two Christoffel symbols, which are necessary for our calculation of the $x$-component of the acceleration, are calculated in Appendix C.

With the help of the obvious equality

$$\frac{dx^\alpha}{ds} = \frac{dx^\alpha}{dt}\frac{dt}{ds}, \tag{29}$$

we can calculate the second derivative of $x^\alpha$ in respect to $s$. Specifically (Straumann, 2013, p. 59),

$$\frac{d^2x^\alpha}{ds^2} = \frac{d}{ds}\left(\frac{dx^\alpha}{dt}\frac{dt}{ds}\right) = \left[\frac{d}{ds}\left(\frac{dx^\alpha}{dt}\right)\right]\frac{dt}{ds} + \frac{dx^\alpha}{dt}\frac{d^2t}{ds^2} = \frac{d^2x^\alpha}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dx^\alpha}{dt}\frac{d^2t}{ds^2}. \tag{30}$$

The second derivative $d^2t/ds^2$, occurring in relation (30), can be calculated by using EoG (28) and relation (29) as

$$\frac{d^2t}{ds^2} \equiv \frac{d^2x^4}{ds^2} = -\Gamma^4_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dt}{ds}\frac{dx^\gamma}{dt}\frac{dt}{ds}. \tag{31}$$

Now, we can calculate the acceleration $d^2x/dt^2$. Identifying $x = x^1$, we obtain

$$\frac{d^2x^1}{ds^2} = -\Gamma^1_{\alpha\beta}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \tag{32}$$

from EoG (28). Supplying relations (30) and (29) into the left-hand and right-hand sides of the last equation, respectively, this equation acquires the form

$$\frac{d^2x^1}{dt^2}\left(\frac{dt}{ds}\right)^2+\frac{dx^1}{dt}\left(-\Gamma^4_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dt}{ds}\frac{dx^\gamma}{dt}\frac{dt}{ds}\right)=-\Gamma^1_{\beta\gamma}\frac{dx^\beta}{dt}\frac{dt}{ds}\frac{dx^\gamma}{dt}\frac{dt}{ds}. \tag{33}$$

Further, if it is divided by $(dt/ds)^2$ and the terms are re-arranged, the $x$-component of acceleration becomes

$$\frac{d^2x^1}{dt^2}=\left(\Gamma^4_{\beta\gamma}\frac{dx^1}{dt}-\Gamma^1_{\beta\gamma}\right)\frac{dx^\beta}{dt}\frac{dx^\gamma}{dt}. \tag{34}$$

For the system of TP and MP in rest, $dx^1/dt\equiv dx/dt=0$, $dx^2/dt\equiv dy/dt=0$, $dx^3/dt\equiv dz/dt=0$, and

$$\frac{dx^4}{dt}\equiv\frac{d\tau}{dt}=\frac{d(ct)}{dt}=c. \tag{35}$$

When we also take into account relation (C17) derived in Appendix C, relation (34) is reduced to

$$\frac{d^2x^1}{dt^2}\equiv\frac{d^2x}{dt^2}=-c^2\Gamma^x_{\tau\tau}=\frac{c^2}{2}\left(g^{xx}\frac{\partial g_{\tau\tau}}{\partial x}+g^{xy}\frac{\partial g_{\tau\tau}}{\partial y}+g^{xz}\frac{\partial g_{\tau\tau}}{\partial z}\right). \tag{36}$$

The contravariant components of the metric tensor in this relation are given by relations (B40)−(B42) derived in Appendix B. With the help of them, the acceleration can be given more explicitly as

$$\begin{aligned}\frac{d^2x}{dt^2}=\frac{c^2}{2}\left\{\left[\frac{x^2}{R^2}\left(1+\frac{1}{g_{rr}}\right)-1\right]\frac{\partial g_{\tau\tau}}{\partial x}+\frac{xy}{R^2}\left(1+\frac{1}{g_{rr}}\right)\frac{\partial g_{\tau\tau}}{\partial y}+\right.\\ \left.+\frac{xz}{R^2}\left(1+\frac{1}{g_{rr}}\right)\frac{\partial g_{\tau\tau}}{\partial z}\right\}.\end{aligned} \tag{37}$$

We assume that both TP and MP are situated in vacuum, therefore the metrics generated by the MP is the OSM. It is characterized by the metric tensor

$$g_{\mu\nu}=\begin{pmatrix} g_{rr} & 0 & 0 & 0\\ 0 & g_{\vartheta\vartheta} & 0 & 0\\ 0 & 0 & g_{\varphi\varphi} & 0\\ 0 & 0 & 0 & g_{\tau\tau}\end{pmatrix}=$$

$$=\begin{pmatrix} -\left(1-\frac{2L}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & -r^2 & 0 & 0\\ 0 & 0 & -r^2\sin^2\vartheta & 0\\ 0 & 0 & 0 & Kc^2\left(1-\frac{2L}{r}\right)\end{pmatrix} \tag{38}$$

Symbols $K$ and $L$ stand for the constants.

In this metrics, the derivatives of component $g_{\tau\tau}$ in respect to spatial coordinates $x$, $y$, and $z$ are

$$\frac{\partial g_{\tau\tau}}{\partial x} = K\frac{2L}{R^3}x, \tag{39}$$

$$\frac{\partial g_{\tau\tau}}{\partial y} = K\frac{2L}{R^3}y, \tag{40}$$

$$\frac{\partial g_{\tau\tau}}{\partial z} = K\frac{2L}{R^3}z, \tag{41}$$

$$\tag{42}$$

therefore formula (37) can further be simplified to

$$\frac{d^2x}{dt^2} = \frac{Kc^2L}{R^3}\left\{\left[\frac{x^2}{R^2}\left(1+\frac{1}{g_{rr}}\right)-1\right]x + \frac{xy^2}{R^2}\left(1+\frac{1}{g_{rr}}\right) + \right.$$
$$\left. + \frac{xz^2}{R^2}\left(1+\frac{1}{g_{rr}}\right)\right\} = \frac{Kc^2L}{R^3}\left[\frac{x^2+y^2+z^2}{R^2}\left(1+\frac{1}{g_{rr}}\right)x - x\right]. \tag{43}$$

Or, after a further handling with this relation, we can obtain

$$\frac{d^2x}{dt^2} = \frac{Kc^2L}{R^3 g_{rr}}x. \tag{44}$$

When the $g_{rr}$ component in the OSM is also expressed explicitly (see relation (38)) the formula for the acceleration in the direction of the coordinate axis $x$ is

$$\frac{d^2x}{dt^2} = -\left(1-\frac{2L}{R}\right)\frac{Kc^2L}{R^3}x. \tag{45}$$

(The formulas for the remaining spatial components of the acceleration, in $y$ and $z$ directions, can simply be obtained by doing the cyclic interchange of variables: $x \to y \to z \to x$.)

When the TP is situated on the coordinate $x$-axis, i.e. the components $y$ and $z$ of its radius vector are zero, then $R = \sqrt{x^2+0+0} = |x|$ and

$$\frac{d^2x}{dt^2} = -\left(1-\frac{2L}{|x|}\right)\frac{c^2KL}{x^2}\frac{x}{|x|}. \tag{46}$$

## 4. Calibration of constants and discussion

Our final formula (45), giving the acceleration of TP in the direction of the coordinate $x$-axis, can be applied to real objects if the constants $K$ and $L$ are determined. Constant $K$ used to be chosen to equal unity. We have kept this constant explicitly up to this point because there are some solutions of the field equations requiring $K \neq 1$.

Let us now establish a new constant, $M$, defined by the relation

$$L = \frac{GM}{c^2}, \tag{47}$$

where $G$ is the Newton gravitational constant. With the help of the new constant and putting $K = 1$, formula (45) can be re-written as

$$\frac{d^2x}{dt^2} = -\left(1 - \frac{2GM}{c^2R}\right) \frac{GM}{R^2} \frac{x}{R}. \tag{48}$$

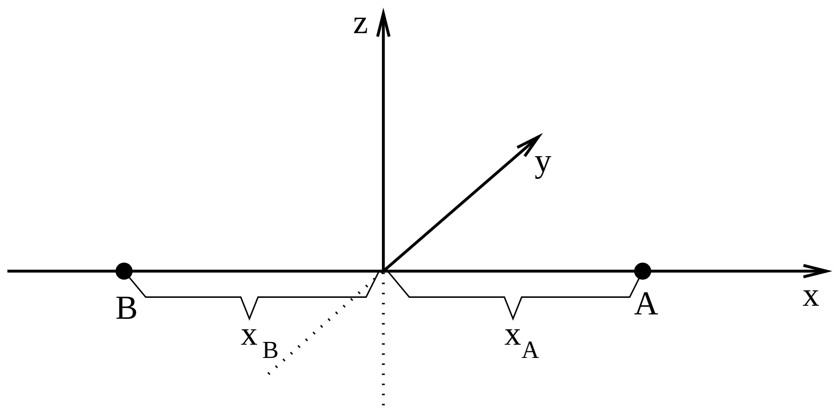


**Figure 1.** The positions, A and B, of the test particle (TP) in the field of a massive, point-like particle, which is placed in the origin of the rectangular coordinate frame. Notice that $x > 0$ at position A, but $x < 0$ at position B.

In the limit of weak gravitational field, i.e. when $2GM/(c^2R) \ll 1$, the last formula is reduced to its counterpart in the Newtonian physics,

$$\frac{d^2x}{dt^2} = -\frac{GM}{x^2} \frac{x}{R}. \tag{49}$$

with $M$ being the mass of the MP. We can see that it correctly determines the orientation of the acceleration. This is clear when we consider two positions of the TP, A and B, on the coordinate $x$-axis shown in Fig. 1. If the TP is at position A, then its $x$ coordinate $x_A > 0$. Hence, $x_A = |x_A|$ and the acceleration $d^2x/dt^2 = -GM/|x_A|^2 < 0$. This means, it is oriented in the opposite direction in respect to the orientation of the coordinate $x$-axis. If the TP is at position B, then $x_B < 0$ and $x_B = -|x_B|$. This implies that the acceleration $d^2x/dt^2 = +GM/|x_B|^2 > 0$. Its orientation is the same as the orientation of the $x$-axis. Thus, seeing Fig. 1, the TP is attracted toward the MP at both positions.

Let us return to the relativistic formula (48). We see that the acceleration is expressed by two terms, explicitly

$$\frac{d^2x}{dt^2} = -\frac{GM}{R^2}\frac{x}{R} + \frac{2G^2M^2}{c^2R^4}x. \tag{50}$$

Notice that the sign of the second, relativistic, term is opposite to the first, Newtonian, term. In other words, the first (second) term is negative (positive) when $x > 0$ and vice versa when $x < 0$. The relativistic term reduces the Newtonian term above the event horizon, i.e. when $2GM/(c^2R) < 1$. When the gravitational acceleration in this region is calculated (i) using the relativistic formula (48) and (ii) the Newtonian formula (49), then the relativistic acceleration is smaller than its Newtonian counterpart.

The relativistic term of acceleration can be understood as a repulsive contribution to the attractive Newtonian acceleration. There have been published solutions of the field equations to model the relativistic compact objects, which implied an outward oriented gravitational attraction in the innermost region of the objects (Ni, 2011; Neslušan, 2015, 2017a,b, 2019; deLyra, 2021; Neslušan, 2022; deLyra & Carneiro, 2023; deLyra et al., 2023). Since the relativistic term is proportional to the mass of MP squared, there can occur a distribution of matter where the relativistic term dominates even above the event horizon (e.g. Neslušan, 2019, Sect. 3). When we take into account this possibility, the outward oriented gravitational attraction in the innermost region of compact objects, according to some models of these, is not surprising.

The relativistic term clearly dominates below the event horizon, i.e. in the regime of gravity with $2GM/(c^2R) > 1$ according to the derived formulas. This means that a TP is repelled from a massive object (black hole?) in this region.

## 5. Conclusion

The orientation of the gravitational attraction within general relativity is more transparent in the O($x$,$y$,$z$,$ct$) coordinate frame than in the Schwarzschild coordinates O($r, \vartheta, \varphi, ct$). The character of the constants in the OSM can also be better discerned in the former frame.

We derived the formula giving the acceleration of a TP in the vicinity of an MP when both particles are in rest. According to our result, the relativistic acceleration consists of two terms implying partial accelerations in mutually opposite directions. The first term is related to the attractive gravity and it is identical with the gravitational acceleration derived from the Newton gravitational law. The second term, the size of which is smaller than the size of the first one above the event horizon (except for some special distributions of matter), implies a certain reduction of the acceleration in a simple case of the gravitational action between two objects. The second term can also be represented as the relativistic, repulsive, complement of the Newtonian formula.

**Acknowledgements.** This work has been supported by VEGA - the Slovak Grant Agency for Science, grant No. 2/0009/22.

## References

deLyra, J. L., Energetic Stability of the Solutions of the Einstein Field Equations for Spherically Symmetric Liquid Shells. 2021, *arXiv e-prints*, arXiv:2101.07214, DOI: 10.48550/arXiv.2101.07214

deLyra, J. L. & Carneiro, C. E. I., Complete solution of the Einstein field equations for a spherical distribution of polytropic matter. 2023, *General Relativity and Gravitation*, **55**, 67, DOI: 10.1007/s10714-023-03115-6

deLyra, J. L., de A. Orselli, R., & Carneiro, C. E. I., Exact solution of the Einstein field equations for a spherical shell of fluid matter. 2023, *General Relativity and Gravitation*, **55**, 68, DOI: 10.1007/s10714-023-03116-5

Einstein, A., Die Feldgleichungen der Gravitation. 1915, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 844

Einstein, A., Die Grundlage der allgemeinen Relativitätstheorie. 1916, *Annalen der Physik*, **354**, 769, DOI: 10.1002/andp.19163540702

Neslušan, L., The Ni's Solution for Neutron Star and Outward Oriented Gravitational Attraction in Its Interior. 2015, *Journal of Modern Physics*, **6**, 2164, DOI: 10.4236/jmp.2015.615220

Neslušan, L., A solution to unconstrained Einstein's equations for a relativistic radiation sphere. in , *New Frontiers in Black Hole Astrophysics*, ed. A. Gomboc, Vol. **324**, 355–356

Neslušan, L., Outline of the concept of stable relativistic radiation sphere. A model of quasar? 2017b, *Astrophysics and Space Science*, **362**, 48, DOI: 10.1007/s10509-017-3027-x

Neslušan, L., The second rise of general relativity in astrophysics. 2019, *Modern Physics Letters A*, **34**, 1950244, DOI: 10.1142/S0217732319502444

Neslušan, L., A demonstration of the difference between the normalized and normalized and non-limited solutions of the field equations in the modeling of relativistic compact objects. 2022, in *Cosmology on Small Scales 2022. Dark Energy and the Local Hubble Expansion Problem*, ed. M. Křížek & Y. V. Dumin (Institute of Mathematics, Czech Academy of Sciences), 85–102

Ni, J., Solutions without a maximum mass limit of the general relativistic field equations for neutron stars. 2011, *Science China Physics, Mechanics, and Astronomy*, **54**, 1304, DOI: 10.1007/s11433-011-4350-9

Schwarzschild, K., Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. 1916, in *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, 424–434

Straumann, N. 2013, *General Relativity* (Springer)

## A. Metric tensor in O($x$,$y$,$z$,$\tau$) frame

Using the general transformation formula (27), the components of the metric tensor, $g_{\alpha,\beta}$, in the O($x$,$y$,$z$,$\tau$) coordinate frame can be calculated and the result is:

$$g_{11} \equiv g_{xx} = \frac{x^2}{R^2} g_{rr} - \frac{x^2 z^2}{R^2 \rho^2} - \frac{y^2}{\rho^2}, \tag{A1}$$

$$g_{12} \equiv g_{xy} = \frac{xy}{R^2} g_{rr} - \frac{xyz^2}{R^2 \rho^2} + \frac{xy}{\rho^2}, \tag{A2}$$

$$g_{13} \equiv g_{xz} = \frac{xz}{R^2} g_{rr} + \frac{xz}{R^2}, \tag{A3}$$

$$g_{14} \equiv g_{x\tau} = 0, \tag{A4}$$

$$g_{22} \equiv g_{yy} = \frac{y^2}{R^2} g_{rr} - \frac{y^2 z^2}{R^2 \rho^2} - \frac{x^2}{\rho^2}, \tag{A5}$$

$$g_{23} \equiv g_{yz} = \frac{yz}{R^2} g_{rr} + \frac{yz}{R^2}, \tag{A6}$$

$$g_{24} \equiv g_{y\tau} = 0, \tag{A7}$$

$$g_{33} \equiv g_{zz} = \frac{z^2}{R^2} g_{rr} - \frac{\rho^2}{R^2}, \tag{A8}$$

$$g_{34} \equiv g_{z\tau} = 0, \tag{A9}$$

$$g_{44} \equiv g_{\tau\tau} = g_{tt}. \tag{A10}$$

Since the product of two derivatives is commutative, i.e. $(\partial x^\alpha/\partial x^\mu)(\partial x^\beta/\partial x^\nu) = (\partial x^\beta/\partial x^\nu)(\partial x^\alpha/\partial x^\mu)$, and $g_{\mu\nu} = g_{\nu\mu}$ in the Schwarzschild coordinates, it is also valid that

$$g_{\alpha\beta} = g_{\beta\alpha}. \tag{A11}$$

In other words, the tensor $g_{\alpha\beta}$ in the O($x$,$y$,$z$,$\tau$) coordinate system can be given as

$$g_{\alpha\beta} = \begin{pmatrix} g_{xx} & g_{xy} & g_{xz} & 0 \\ g_{xy} & g_{yy} & g_{yz} & 0 \\ g_{xz} & g_{yz} & g_{zz} & 0 \\ 0 & 0 & 0 & g_{\tau\tau} \end{pmatrix} \tag{A12}$$

If the TP is located on the coordinate $x$-axis, then $y = z = 0$ and the the spatial components of the metric tensor $g_{\alpha\beta}$ reduce to

$$g_{xx} = g_{rr}, \tag{A13}$$

$$g_{yy} = -1, \tag{A14}$$

$$g_{zz} = -1, \tag{A15}$$

$$g_{\alpha\beta} = 0 \qquad \text{for} \quad \alpha \neq \beta. \tag{A16}$$

## B. Contravariant components of metric tensor

To calculate the Christoffel symbols (in Appendix C), we need to know the contravariant form of the metric tensor $g_{\alpha\beta}$. It is well known that the covariant and contravariant components are related by the formula

$$g_{\alpha\sigma}g^{\sigma\beta} = \delta_{\alpha}^{\beta}, \tag{B1}$$

where $\delta_{\alpha}^{\beta}$ is the Kronecker delta (it is equal to 1 if $\alpha = \beta$ and 0 if $\alpha \neq \beta$).

When we use relation (B1), the fact that $g_{\alpha\beta} = g_{\beta\alpha}$, and omit zero terms, we can write the following five sets of equations:
1-st set:

$$g_{11}g^{11} + g_{12}g^{21} + g_{13}g^{31} = 1, \tag{B2}$$
$$g_{12}g^{11} + g_{22}g^{21} + g_{23}g^{31} = 0, \tag{B3}$$
$$g_{13}g^{11} + g_{23}g^{21} + g_{33}g^{31} = 0, \tag{B4}$$
$$\tag{B5}$$

2-nd set:

$$g_{11}g^{12} + g_{12}g^{22} + g_{13}g^{32} = 0, \tag{B6}$$
$$g_{12}g^{12} + g_{22}g^{22} + g_{23}g^{32} = 1, \tag{B7}$$
$$g_{13}g^{12} + g_{23}g^{22} + g_{33}g^{32} = 0, \tag{B8}$$
$$\tag{B9}$$

3-rd set:

$$g_{11}g^{13} + g_{12}g^{23} + g_{13}g^{33} = 0, \tag{B10}$$
$$g_{12}g^{13} + g_{22}g^{23} + g_{23}g^{33} = 0, \tag{B11}$$
$$g_{13}g^{13} + g_{23}g^{23} + g_{33}g^{33} = 1, \tag{B12}$$
$$\tag{B13}$$

4-th set:

$$g_{11}g^{14} + g_{12}g^{24} + g_{13}g^{34} = 0, \tag{B14}$$
$$g_{12}g^{14} + g_{22}g^{24} + g_{23}g^{34} = 0, \tag{B15}$$
$$g_{13}g^{14} + g_{23}g^{24} + g_{33}g^{34} = 0, \tag{B16}$$
$$\tag{B17}$$

5-th set:

$$g_{44}g^{41} = 0, \tag{B18}$$
$$g_{44}g^{42} = 0, \tag{B19}$$
$$g_{44}g^{43} = 0, \tag{B20}$$
$$g_{44}g^{44} = 1. \tag{B21}$$

Since $g_{44} \neq 0$, Set 5 immediately implies $g^{41} = g^{42} = g^{43} = 0$ and $g^{44} = 1/g_{44}$. Each of Sets 1 to 4 is the system of three equations with three unknown variables. We can use the method of determinants to solve each system. The basic determinant is the same for all four sets,

$$D = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{vmatrix} \equiv \begin{vmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{xy} & g_{yy} & g_{yz} \\ g_{xz} & g_{yz} & g_{zz} \end{vmatrix} =$$

$$= g_{xx}g_{yy}g_{zz} + 2g_{xy}g_{xz}g_{yz} - g_{xz}^2 g_{yy} - g_{xy}^2 g_{zz} - g_{xx}g_{yz}^2. \tag{B22}$$

After we solve the system of equations, we obtain:
from Set 1:

$$g^{11} \equiv g^{xx} = \frac{1}{D}(g_{yy}g_{zz} - g_{yz}^2), \tag{B23}$$

$$g^{21} \equiv g^{yx} = \frac{1}{D}(g_{xz}g_{yz} - g_{xy}g_{zz}), \tag{B24}$$

$$g^{31} \equiv g^{zx} = \frac{1}{D}(g_{xy}g_{yz} - g_{xz}g_{yy}); \tag{B25}$$

from Set 2:

$$g^{12} \equiv g^{xy} = \frac{1}{D}(g_{xz}g_{yz} - g_{xy}g_{zz}), \tag{B26}$$

$$g^{22} \equiv g^{yy} = \frac{1}{D}(g_{xx}g_{zz} - g_{xz}^2), \tag{B27}$$

$$g^{32} \equiv g^{zy} = \frac{1}{D}(g_{xy}g_{xz} - g_{xx}g_{yz}); \tag{B28}$$

from Set 3:

$$g^{13} \equiv g^{xz} = \frac{1}{D}(g_{xy}g_{yz} - g_{xz}g_{yy}), \tag{B29}$$

$$g^{23} \equiv g^{yz} = \frac{1}{D}(g_{xy}g_{xz} - g_{xx}g_{yz}), \tag{B30}$$

$$g^{33} \equiv g^{zz} = \frac{1}{D}(g_{xx}g_{yy} - g_{xy}^2); \tag{B31}$$

and from Set 4: $g^{14} \equiv g^{x\tau} = 0$, $g^{24} \equiv g^{y\tau} = 0$, and $g^{34} \equiv g^{z\tau} = 0$. We see that $g^{\alpha 4} = g^{4\alpha}$.

Let us now derive the determinant $D$ and the contravariant components explicitly. The determinant, given by relation (B22), can be re-written as

$$D = g_{xx}A + g_{xz}B + g_{xy}C, \tag{B32}$$

where we denoted

$$A = g_{xx}g_{zz} - g_{yz}^2, \tag{B33}$$

$$B = g_{xy}g_{yz} - g_{xz}g_{yy}, \tag{B34}$$

$$C = g_{xz}g_{yz} - g_{xy}g_{zz}. \tag{B35}$$

When we supply the explicit forms of the $g_{\alpha\beta}$ components given by relations (A1) to (A8) into the last three relations, we obtain

$$A = \frac{1}{R^2} \left[ x^2 - \left( y^2 + z^2 \right) g_{rr} \right], \tag{B36}$$

$$B = \frac{xz}{R^2} \left( g_{rr} + 1 \right), \tag{B37}$$

$$C = \frac{xy}{R^2} \left( g_{rr} + 1 \right), \tag{B38}$$

and the determinant

$$D = g_{rr} \tag{B39}$$

after some algebraic handling.

When relations (A1) to (A8) and (B39) are supplied into (B23), (B26), and (B29), they acquire the form

$$g^{xx} = \frac{x^2}{R^2} \left( 1 + \frac{1}{g_{rr}} \right) - 1, \tag{B40}$$

$$g^{xy} = \frac{xy}{R^2} \left( 1 + \frac{1}{g_{rr}} \right), \tag{B41}$$

$$g^{xz} = \frac{xz}{R^2} \left( 1 + \frac{1}{g_{rr}} \right). \tag{B42}$$

For a particle situated on the coordinate $x$-axis, we can prove that the only non-zero contravariant components are

$$g^{11} \equiv g^{xx} = \frac{1}{g_{xx}} = \frac{1}{g_{rr}}, \tag{B43}$$

$$g^{22} \equiv g^{yy} = \frac{1}{g_{yy}} = -1, \tag{B44}$$

$$g^{33} \equiv g^{zz} = \frac{1}{g_{zz}} = -1, \tag{B45}$$

$$g^{44} \equiv g^{\tau\tau} = \frac{1}{g_{\tau\tau}}. \tag{B46}$$

## C. Christoffel symbols $\Gamma^1_{\beta\gamma}$ and $\Gamma^4_{\beta\gamma}$

We recall the well-known general formula to calculate the Christoffel symbols,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left( \frac{\partial g_{\sigma\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\sigma\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}} \right). \tag{C1}$$

In the following, this formula is used to calculate the symbols figuring in relation (34) that we use in our derivation. At first, we give the implicit result.

The symbols with the upper index "1" are

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^1} + g^{12} \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2} g^{12} \frac{\partial g_{11}}{\partial x^2} + g^{13} \frac{g_{13}}{\partial x^1} - \frac{1}{2} g^{13} \frac{\partial g_{11}}{\partial x^3}, \tag{C2}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^2} + \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^1} + \frac{1}{2} g^{13} \left( \frac{\partial g_{13}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^3} \right), \tag{C3}$$

$$\Gamma^1_{13} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^3} + \frac{1}{2} g^{12} \left( \frac{\partial g_{12}}{\partial x^3} + \frac{\partial g_{23}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3} \right) + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial x^1}, \tag{C4}$$

$$\Gamma^1_{14} = 0, \tag{C5}$$

$$\Gamma^1_{21} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^2} + \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^1} + \frac{1}{2} g^{13} \left( \frac{\partial g_{23}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^3} \right), \tag{C6}$$

$$\Gamma^1_{22} = g^{11} \frac{\partial g_{12}}{\partial x^2} - \frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1} + \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^2} + g^{13} \frac{\partial g_{23}}{\partial x^2} - \frac{1}{2} g^{13} \frac{\partial g_{22}}{\partial x^3}, \tag{C7}$$

$$\Gamma^1_{23} = \frac{1}{2} g^{11} \left( \frac{\partial g_{12}}{\partial x^3} + \frac{\partial g_{13}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^1} \right) + \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^3} + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial x^2}, \tag{C8}$$

$$\Gamma^1_{24} = 0, \tag{C9}$$

$$\Gamma^1_{31} = \frac{1}{2} g^{11} \frac{\partial g_{11}}{\partial x^3} + \frac{1}{2} g^{12} \left( \frac{\partial g_{23}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^2} \right) + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial x^1}, \tag{C10}$$

$$\Gamma^1_{32} = \frac{1}{2} g^{11} \left( \frac{\partial g_{13}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^1} \right) + \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^3} + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial x^2}, \tag{C11}$$

$$\Gamma^1_{33} = g^{11} \frac{\partial g_{13}}{\partial x^3} - \frac{1}{2} g^{11} \frac{\partial g_{33}}{\partial x^1} + g^{12} \frac{\partial g_{23}}{\partial x^3} - \frac{1}{2} g^{12} \frac{\partial g_{33}}{\partial x^2} + \frac{1}{2} g^{13} \frac{\partial g_{33}}{\partial x^3}, \tag{C12}$$

$$\Gamma^1_{34} = 0, \tag{C13}$$

$$\Gamma^1_{41} = 0, \tag{C14}$$

$$\Gamma^1_{42} = 0, \tag{C15}$$

$$\Gamma^1_{43} = 0, \tag{C16}$$

$$\Gamma^1_{44} = -\frac{1}{2} g^{11} \frac{\partial g_{44}}{\partial x^1} - \frac{1}{2} g^{12} \frac{\partial g_{44}}{\partial x^2} - \frac{1}{2} g^{13} \frac{\partial g_{44}}{\partial x^3}. \tag{C17}$$

Notice that $\Gamma^1_{\alpha\beta} = \Gamma^1_{\beta\alpha}$.

The non-zero symbols with the upper index "4" are

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{1}{2} g^{44} \frac{\partial g_{44}}{\partial x^1}, \tag{C18}$$

$$\Gamma^4_{24} = \Gamma^4_{42} = \frac{1}{2} g^{44} \frac{\partial g_{44}}{\partial x^2}, \tag{C19}$$

$$\Gamma^4_{34} = \Gamma^4_{43} = \frac{1}{2} g^{44} \frac{\partial g_{44}}{\partial x^3}. \tag{C20}$$

When the TP and MP are in rest, we need to know, explicitly, only symbol $\Gamma^1_{44} \equiv \Gamma^x_{\tau\tau}$ to calculate the $x$-component of the acceleration. In the simplest case, when the TP is situated on the coordinate $x$-axis, i.e. $y = z = 0$, we found, in Appendix B, that $g^{xy} = g^{xz} = 0$. Using these expressions and relation (B43), relation (C17) reduces to

$$\Gamma^1_{44} \equiv \Gamma^x_{\tau\tau} = -\frac{1}{2g_{xx}} \frac{\partial g_{\tau\tau}}{\partial x}. \tag{C21}$$